

# Idempotents with polynomial coefficients

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## Abstract

We combine Young idempotents in the group algebra of the symmetric group with the action of the symmetric group on products of Vandermonde determinants to obtain idempotents with polynomial coefficients.

*Key words:* Young idempotents, symmetric group

Young described a complete set of idempotents in the group algebra of the symmetric group. However, given an explicit idempotent, how does one connect it with Young's work?

In this text, I take the example of (quasi)-idempotents of the type

$$\sum_{\sigma \in \mathfrak{S}_n} (x_{\sigma_1} - x_{\sigma_2}) \sigma$$

in the group algebra of the symmetric group  $\mathfrak{S}_n$  with polynomial coefficients.

Given a list of integers  $\alpha = [\alpha_1, \dots, \alpha_m]$ , let

$$\Delta(\alpha) = \prod_{1 \leq i \leq j \leq m} (x_{\alpha_i} - x_{\alpha_j}).$$

Given a partition  $\lambda = [\lambda_1, \dots, \lambda_r]$  of weight  $n = |\lambda| = \lambda_1 + \dots + \lambda_r$ , and a permutation  $\sigma \in \mathfrak{S}_n$ , define

$$\Delta(\sigma) = \Delta(\sigma_1, \dots, \sigma_{\lambda_1}) \Delta(\sigma_{\lambda_1+1}, \dots, \sigma_{\lambda_1+\lambda_2}) \dots \Delta(\sigma_{\lambda_1+\dots+\lambda_{r-1}+1}, \dots, \sigma_{\lambda_1+\dots+\lambda_r}),$$

which corresponds to cutting  $\sigma$  into blocks of respective lengths  $\lambda_1, \lambda_2, \dots$ .

We shall consider the element

$$\Omega_\lambda := \sum_{\sigma \in \mathfrak{S}_n} \Delta_\lambda(\sigma) \sigma \in \mathfrak{Pol}(x_1, \dots, x_n)[\mathfrak{S}_n],$$

and show that this element is quasi-idempotent, i.e. that there exists a (non-zero) polynomial  $c_\lambda$  such that  $\Omega_\lambda \Omega_\lambda = c_\lambda \Omega_\lambda$ .

Let us evoke in a few words the work of Young. Young described a linear basis  $\{t_{tu}\}$  of the group algebra  $\mathbb{C}[\mathfrak{S}_n]$ , indexed by pairs of standard tableaux  $t, u$  of the same shape of  $n$  boxes. These  $n!$  elements satisfy

$$e_{t,u}e_{v,w} = \delta_{u,v} e_{t,w}.$$

Young basis can be characterized as the family of elements in  $\mathbb{C}[\mathfrak{S}_n]$  which are left and right eigenfunctions of the Jucys-Murphy elements [5, 1].

The central idempotent corresponding to a partition  $\lambda$  of  $n$  is  $e_\lambda = \sum_t e_{t,t}$ , sum over all standard tableaux of shape  $\lambda$ . In particular, when  $\lambda$  has only one part, the idempotent  $e_n = (n!)^{-1} \sum \sigma$  is called the *trivial idempotent*.

The unit of  $\mathbb{C}[\mathfrak{S}_n]$  decomposes as  $\sum_{\lambda: |\lambda|=n} e_\lambda = \sum_t e_{t,t}$ , sum over all standard tableaux. For example, for  $n = 2$ , there are two tableaux and the corresponding decomposition is  $1 = \frac{1}{2}(1+s_1) + \frac{1}{2}(1-s_1)$ . We more generally write  $s_1, \dots, s_{n-1}$  for the *simple transpositions*, which generate  $\mathfrak{S}_n$ . More generally, given a partition  $\lambda$ , the group  $\mathfrak{S}_\lambda$  generated by all  $s_i$ ,  $i \notin \{\lambda_1, \lambda_1+\lambda_2, \lambda_1+\lambda_2+\lambda_3, \dots\}$  is called a *Young subgroup*.

Given a partition  $\lambda$ , let  $\mathfrak{Tab}(\lambda)$  be the set of standard tableaux of columns of respective lengths  $\lambda_1, \lambda_2, \dots$ . The *first tableau*  $\clubsuit$  is the tableau filled by consecutive integers in each column, and similarly, for the *last tableau*  $\spadesuit$  one requires consecutive integers in each row.

Idempotents  $e_{t,t}$  corresponding to the same shape are obtained from any of them by conjugation by factors of the type  $(s_i + c)$ .

For example, for the shape  $[2, 2, 1]$ , the five idempotents are obtained recursively from the first idempotent  $e_{\clubsuit, \clubsuit}$ , according to the following graph.

$$\begin{array}{c}
\clubsuit = \begin{array}{|c|c|} \hline 3 & \\ \hline 2 & 5 \\ \hline 1 & 4 \\ \hline \end{array} \\
\Downarrow \\
\begin{array}{|c|c|} \hline 4 & \\ \hline 2 & 5 \\ \hline 1 & 3 \\ \hline \end{array} \\
(s_3 - \frac{1}{3}) \frac{e_{\clubsuit, \clubsuit}}{1 - \frac{1}{9}} (s_3 - \frac{1}{3}) \\
\swarrow \quad \searrow \\
\begin{array}{|c|c|} \hline 4 & \\ \hline 3 & 5 \\ \hline 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 5 & \\ \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \\
(s_2 - \frac{1}{2})(s_3 - \frac{1}{3}) \frac{e_{\clubsuit, \clubsuit}}{(1 - \frac{1}{9})(1 - \frac{1}{4})} (s_3 - \frac{1}{3})(s_2 - \frac{1}{2}) & (s_4 - \frac{1}{2})(s_3 - \frac{1}{3}) \frac{e_{\clubsuit, \clubsuit}}{(1 - \frac{1}{9})(1 - \frac{1}{4})} (s_3 - \frac{1}{3})(s_4 - \frac{1}{2}) \\
\swarrow \quad \searrow \\
\spadesuit = \begin{array}{|c|c|} \hline 5 & \\ \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} \\
e_{\spadesuit, \spadesuit} = (s_2 - \frac{1}{2})(s_4 - \frac{1}{2})(s_3 - \frac{1}{3}) \frac{e_{\clubsuit, \clubsuit}}{(1 - \frac{1}{9})(1 - \frac{1}{4})(1 - \frac{1}{4})} (s_3 - \frac{1}{3})(s_4 - \frac{1}{2})(s_2 - \frac{1}{2})
\end{array}$$

The general rule to write the factors  $(s_i + c)$  translates in terms of group algebra Young's matrices of representation [1]. Define the *content*  $c(i)$  of the integer  $i$  located in the  $j$ -th column and  $k$ -th row of a tableau  $t$  to be  $j - k$ . Then, given any  $i$  such that the image  $s_i t$  of  $t$  under the exchange of  $i$  and  $i+1$  is still a tableau, one has, for any  $u$  of the same shape,

$$\begin{aligned}
e_{s_i t, u} &= \frac{1}{\sqrt{1 - \frac{1}{(c(i) - c(i+1))^2}}} \left( s_i + \frac{1}{c(i) - c(i+1)} \right) e_{t, u} \\
e_{u, s_i t} &= e_{u, t} \left( s_i + \frac{1}{c(i) - c(i+1)} \right) \frac{1}{\sqrt{1 - \frac{1}{(c(i) - c(i+1))^2}}}
\end{aligned}$$

For a pair of tableaux  $t, u$  in  $\mathfrak{Tab}(\lambda)$ , let  $s_i s_j \cdots s_k$  be a product (it does to need to be of minimum length) such that  $s_i s_j \cdots s_k u = t$ , denote by  $\sigma_{t, u}$  this product, and by  $\zeta(t, u)$  the corresponding product of factors  $\frac{1}{\sqrt{1 - \frac{1}{(c(i) - c(i+1))^2}}} \left( s_i + \frac{1}{c(i) - c(i+1)} \right)$ . Beware that the contents are relative to variable tableaux in a sequence. The above products are better understood when using Yang-Baxter graphs and spectral vectors [4].

Given  $\lambda$ , let  $Y_{\clubsuit} = \Delta_{\lambda}(1, \dots, n)$ . The module  $V_{\lambda}$  generated by the action of  $\mathfrak{S}_n^x$  on  $Y_{\clubsuit}$  is irreducible and called *Specht module*[2]. The *Specht basis* is  $\{\sigma_{t,\clubsuit}^x Y_{\clubsuit}, t \in \mathfrak{Tab}(\lambda)\}$ , while the Young basis is

$$\{Y_t := e_{t,\clubsuit}^x Y_{\clubsuit} = \zeta(t, \clubsuit) Y_{\clubsuit}, t \in \mathfrak{Tab}(\lambda)\}.$$

Given two copies  $\mathfrak{S}_n^1, \mathfrak{S}_n^2$  of  $\mathfrak{S}_n$ , one defines the *diagonal group*  $\mathfrak{S}_n^{12}$  to be the other copy of  $\mathfrak{S}_n$  generated by  $s_1^1 s_1^2, \dots, s_{n-1}^1 s_{n-1}^2$ .

The trivial diagonal idempotent  $e_n^{12} = (n!)^{-1} \sum \sigma^{12}$  is such that, for any  $\sigma \in \mathfrak{S}_n$ , one has

$$e_n^{12} \sigma^1 = e_n^{12} (\sigma^2)^{-1} \quad \& \quad \sigma^1 e_n^{12} = (\sigma^2)^{-1} e_n^{12}.$$

Therefore, for any pair  $(t, u)$  of tableaux of the same shape, one has

$$e_{t,t}^1 e_n^{12} e_{u,u}^1 = c e_{t,u}^1 e_{t,u}^2$$

with some non-zero constant  $c$ .

Checking this constant [4] furnishes the following expression of  $e_n^{12}$ , noting as usual  $f_{\lambda}$  the number of standard tableaux in  $\mathfrak{Tab}(\lambda)$  :

$$e_n^{12} = \sum_{\lambda: |\lambda|=n} \frac{1}{f_{\lambda}} \sum_{t,u \in \mathfrak{Tab}(\lambda)} e_{t,u}^1 e_{t,u}^2. \quad (1)$$

Let us now take  $\mathfrak{S}_n^1 = \mathfrak{S}_n$ , the second copy  $\mathfrak{S}_n^2$  being the symmetric group  $\mathfrak{S}_n^x$  acting by permutation on  $x_1, \dots, x_n$ . Denote by  $\tilde{\sigma}$  the elements of the diagonal group  $\tilde{\mathfrak{S}}_n$ . Notice that  $\Omega_{\lambda} = (\sum_{\sigma \in \mathfrak{S}_n} \sigma \sigma^x) Y_{\clubsuit}$ .

The element  $\Omega_{\lambda}$  is by construction invariant under left multiplication by all  $\tilde{s}_i$ . Therefore one has

$$\Omega_{\lambda} = \tilde{e}_n \Omega_{\lambda} = \sum_{t,u \in \mathfrak{Tab}(\lambda)} e_{t,u} e_{t,u}^x \Omega_{\lambda} = \sum_{t,u \in \mathfrak{Tab}(\lambda)} e_{t,u} (e_{t,u}^x Y_{\clubsuit}). \quad (2)$$

However, Specht polynomials in  $V_{\lambda}$  are annihilated by all  $e_{\nu}^x$ ,  $\nu \neq \mu = \lambda^{\sim}$ . Consequently, the sum (2) restricts to

$$\Omega_{\lambda} = \tilde{e}_{\mu} \Omega_{\lambda} = \sum_{t,u \in \mathfrak{Tab}(\lambda)} e_{t,u} (e_{t,u}^x Y_{\clubsuit}). \quad (3)$$

On the other hand,  $\Omega_{\lambda} s_i = -\Omega_{\lambda}$  for all  $s_i$  belonging to the Young subgroup  $\mathfrak{S}_{\lambda}$ , because for those  $s_i$  one has  $\Delta(\sigma s_i) = -\Delta(\sigma s_i)$ . Therefore  $\Omega_{\lambda}$  is

alternating under the Young subgroup  $\mathfrak{S}_\lambda$  acting on its right and in the sum (3), the tableaux  $u$  must coincide with the first tableau  $\clubsuit$ . The sum becomes

$$\Omega_\lambda = \sum_{t \in \mathfrak{Tab}(\lambda)} e_{t, \clubsuit} e_{t, \clubsuit}^x \Omega_\lambda = \frac{n!}{f_\lambda} \sum_{t \in \mathfrak{Tab}(\lambda)} e_{t, \clubsuit} (e_{t, \clubsuit}^x Y_\clubsuit) = \sum_{t \in \mathfrak{Tab}(\lambda)} \frac{n!}{f_\lambda} e_{t, \clubsuit} Y_t. \quad (4)$$

In final, one has obtained the following expression.

**Theorem 1** *Given a partition  $\lambda$  of  $n$ , then the element  $\Omega_\lambda$  decomposes as*

$$\frac{f_\lambda}{n!} \Omega_\lambda = \sum_{t \in \mathfrak{Tab}(\lambda)} Y_t e_{t, \clubsuit} = \left( \sum_{t \in \mathfrak{Tab}(\lambda)} \zeta^x(t, \clubsuit) \zeta(t, \clubsuit) \right) Y_\clubsuit e_{\clubsuit, \clubsuit}. \quad (5)$$

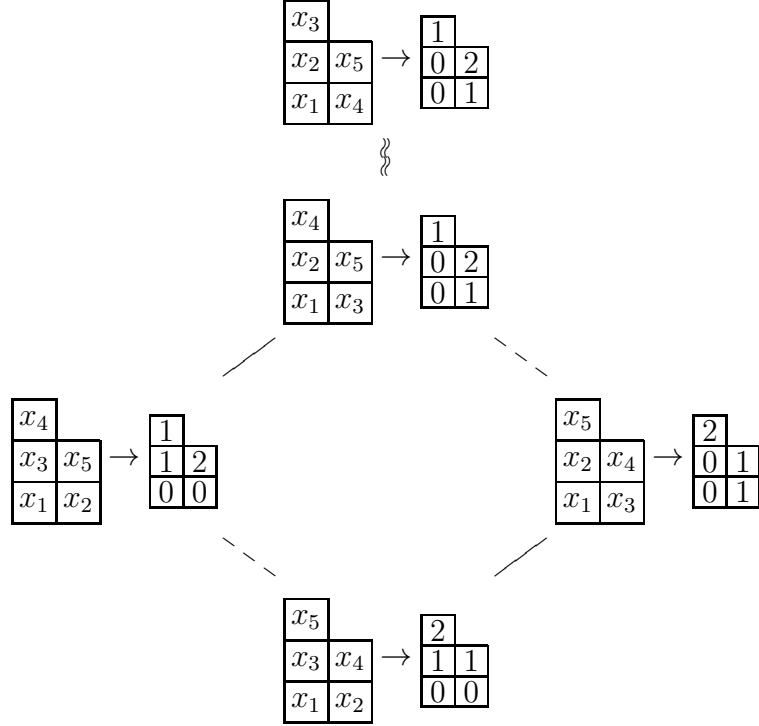
*This is moreover a quasi-idempotent :*

$$\Omega_\lambda \Omega_\lambda = \frac{n!}{f_\lambda} Y_\clubsuit \Omega_\lambda.$$

For example, for  $\lambda = [2, 2]$ , one has

$$\Omega_{2,2} = \frac{4!}{2} \left( \frac{s_2^x - \frac{1}{2}}{\sqrt{1 - \frac{1}{4}}} \frac{s_2 - \frac{1}{2}}{\sqrt{1 - \frac{1}{4}}} + 1 \right) (x_1 - x_2)(x_3 - x_4) e_{\clubsuit, \clubsuit}.$$

Polynomials offer the the possibility of using interpolation methods. For example, let us consider the specialization  $\mathbf{x} = [0^{\mu_1}, 1^{\mu_2}, \dots]$ , i.e.  $x_1 = 0 = \dots = x_{\mu_1}$ ,  $x_{\mu_1+1} = 1 = \dots = x_{\mu_1+\mu_2}, \dots$ . The Specht polynomials corresponding to the standard tableaux of shape  $\mu = \lambda^\sim$ , which are product of Vandermonde functions, vanish uniformly for this specialization, except in the case of the last tableau  $\spadesuit$ . Indeed, there is only one way of distributing the integers  $0^{\mu_1}, 1^{\mu_2}, \dots$  to avoid having two equal entries in the same column of the specialized tableau. For example, for  $\lambda = [3, 2]$ , interpreting each tableau as a product of Vandermonde functions corresponding to its columns, one has the following specializations :



Because of the triangularity of the Young basis with respect to the Specht basis, one obtains the following proposition.

**Proposition 2** *Let  $\lambda$  be a partition of  $n$ ,  $\mu = \lambda^\sim$ . Then the specialization  $\mathbf{x} = [0^{\mu_1}, 1^{\mu_2}, \dots]$  of  $\Omega_\lambda$  is equal to*

$$g_\lambda := (\lambda_1 - 1)! (\lambda_2 - 1)! \cdots \frac{n!}{\lambda} e_{\clubsuit, \clubsuit}.$$

Notice that  $e_{\clubsuit, \clubsuit} e_{\clubsuit, \clubsuit} = 0$ , the specialization of  $\Omega_\lambda$  does not furnish a quasi-idempotent because  $Y_\clubsuit$  specializes to 0. On the other hand,  $\sigma_{\clubsuit, \spadesuit} e_{\clubsuit, \clubsuit}$  is such that

$$(\sigma_{\clubsuit, \spadesuit} e_{\clubsuit, \clubsuit}) (\sigma_{\clubsuit, \spadesuit} e_{\clubsuit, \clubsuit}) = \sigma_{\clubsuit, \spadesuit} e_{\clubsuit, \clubsuit}.$$

Continuing with the same example, the element  $\frac{1}{2}\Omega_{32}$  specializes into  $12e_{\clubsuit, \clubsuit}$  and can be factorized into

$$g_{3,2} = s_2 s_4 s_3 (1 + s_1 s_2 s_3 + s_2 s_4 s_3 + s_2 s_3 s_4 s_1 s_2 s_3) \sum_{\sigma \in \mathfrak{S}_{3,2}} (-1)^{\ell(\sigma)} \sigma.$$

In that case,  $\sigma_{\clubsuit, \clubsuit} = s_3 s_2 s_4$  and one can check directly that  $s_3 s_2 s_4 g_{3,2}$  is a quasi-idempotent.

The coefficients in the expansion of a generic idempotent are non null, and it is not easy to manipulate sums  $\sum c_\sigma \sigma$  of  $n!$  terms. However, one can transform formally such a sum into a polynomial  $\sum c_\sigma y^{\mathbf{c}(\sigma)}$ , using the *code*  $\mathbf{c}(\sigma)$  of a permutation defined by

$$\mathbf{c}(\sigma) = [c_1, \dots, c_n], \quad , \quad c_i = \#\{j \geq i, \sigma_j \leq \sigma_i\}.$$

We have exhibited in [3], for each partition  $\lambda$ , an element in the isotypic component  $e_\lambda \mathbb{C}[\mathfrak{S}_n] e_\lambda$ , which, as a polynomial in  $\mathbf{y}$ , factorizes into simple factors. In the notation of the present article, this distinguished element is  $\omega g_\lambda \omega$ , with  $\omega = [n, \dots, 1]$ .

For example,  $[5, 4, 3, 2, 1] g_{3,2} [5, 4, 3, 2, 1]$  is sent onto

$$y_1 y_2 (1 + y_1) (1 + y_2) (y_1^2 - y_2) (1 - y_3 + y_3^2) (y_4 - 1)$$

which represents a sum of  $2^4 \times 3$  permutations with coefficients equal to  $\pm 1$ . A more impressive example is the image of  $[7, 6, 5, 4, 3, 2, 1] g_{4,2,1} [7, 6, 5, 4, 3, 2, 1]$ , which is

$$y^{4220000} (1 + y_1 + y_1^2) (1 + y_2) (1 + y_3) (y_2^2 - y_3) (1 - y_4 + y_4^2 - y_4^3) (1 - y_5 + y_5^2) (1 - y_6),$$

which expands into a sum of  $2^4 \times 3 \times 4 = 576$  permutations. We refer to [3] for the precise rule for writing the factors (we have here an extra monomial factor, because we are not using the same normalization as in [3]).

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